APPROXIMATE UNITARY EQUIVALENCE OF FINITE INDEX ENDOMORPHISMS OF THE AFD FACTORS

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Abstract. We consider two finite index endomorphisms ρ, σ of any AFD factor M. We characterize the condition for there being a sequence $\{u_n\}$ of unitaries of the factor M with $\mathrm{Ad}u_n \circ \rho \to \sigma$. The characterization is given by using the canonical extension of endomorphisms, which is introduced by Izumi. Our result is a generalization of the characterization of approximate innerness of endomorphisms of the AFD factors, obtained by Kawahiashi–Sutherland–Takesaki and Masuda–Tomatsu. Our proof, which does not depend on the types of factors, is based on recent development on the Rohlin property of flows on von Neumann algebras.

1. Introduction

In this paper, we characterize the approximate innerness of the difference of two finite index endomorphisms of the AFD factors of type III (Theorem 2). More precisely, for two finite index endomorphisms ρ ,σ of any AFD factor M, we give a good necessary and sufficient condition for there being a sequence $\{u_n\}$ of unitaries of M with $\mathrm{Ad}u_n \circ \rho \to \sigma$ as $n \to \infty$ in the sense of Masuda–Tomatsu [15]. First of all, we explain the reason why we are interested in this topic. The reason is that this result should be useful for classifying group actions. It has been important to classify group actions on von Neumann algebras up to cocycle conjugacy. Since a remarkable work of Connes [3], classification of group actions on the AFD factors has greatly been developed by many researchers. In particular, the actions of discrete amenable groups on the AFD factors are completely classified (See Jones [7], Ocneanu [20], Sutherland–Takesaki [22], Kawahigashi–Takesaki–Sutherland [11] and Katayama–Sutherland–Takesaki [10]). It is interesting to note that although there are many different actions up to conjugacy, they are clearly classified when we ignore the difference of cocycle conjugacy. One of the next problems is to classify actions of continuous groups. Among them, classification of actions of compact groups is considered to be relatively easy because the dual of a compact group is discrete. In fact, actions of compact abelian groups on the AFD factors have completely been classified by using this observation (See Jones–Takesaki [8] and Kawahigashi–Takesaki [12]). However, when it comes to classifying actions of non-abelian compact groups, the problem is much more difficult. One of the reasons is that the dual action of an action of a non-abelian compact group is a collection of endomorphisms, not of automorphisms. Hence in order to proceed with classifying actions, it is important to understand the properties of endomorphisms. In the proof of classification theorems of group actions, whether the difference of two actions is approximated by inner automorphisms or not is very important. Hence we should characterize the approximate innerness of the difference of two endomorphisms of the AFD factors.

In this paper, we characterize the approximate innerness of the difference of two endomorphisms in the sense of Masuda–Tomatsu [15] (Theorem 2). In Masuda–Tomatsu [16], they propose a conjecture of the complete invariant for actions of discrete Kac algebras on the AFD factors (Conjecture 8.2 of [17]). The dual of minimal actions of compact groups are ones of them. Our main theorem implies that if two actions of discrete Kac algebras on the AFD factors of type III have the same invariants, the difference of these two actions is approximately inner (See Problem 8.3 and the preceding argument to that problem of [16]). Our main theorem characterizes when one endomorphism transits to another endomorphism. Hence the theorem may also be seen as a kind of endomorphism counterpart of the main theorem of Haagerup–Størmer [5], which characterizes when one normal state of a von Neumann algebra transits to another normal state. It is important to note that our main theorem is a generalization of Theorem 1 (1) of Kawahigashi–Sutherland–Takesaki [11] and Theorem 3.15 of Masuda–Tomatsu [15]. The proof of our theorem is based on recent development on the Rohlin property of flows on von Neumann algebras, which does not depend on the types of the AFD factors. Our method is also applicable to the characterization of the central triviality of automorphisms (Theorem 1. (2) of Kawahigashi–Sutherland–Takesaki [11]). In appendix, we give another proof of the characterization of the central triviality, which does not depend on the types of the AFD factors.

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2. Preliminaries

2.1. **Notations.** Let M be a von Neumann algebra. For a normal positive linear functional ψ of M and $x \in M$, set

$$\|x\|_\psi := \sqrt{\psi(x^*x)},$$
$$\|x\|_\psi^\sharp := \sqrt{\frac{\psi(x^*x)+\psi(xx^*)}{2}}.$$

Lemma 1. *Let λ be a σ-weakly continuous linear functional of a von Neumann algebra M and $\lambda = \psi v$ be its polar decomposition. Then we have*

$$\|\lambda a\| \leq \psi(vaa^*v^*)^{1/2}\|\lambda\|^{1/2},$$
$$\|a\lambda\| \leq \psi(a^*a)^{1/2}\|\lambda\|^{1/2}$$

for any $a \in M$.

Proof. By Cauchy–Schwarz's inequality, for $x \in M$, we have

$$\begin{aligned}|\lambda a(x)| &= |\psi(vax)| \\ &\leq \psi(vaa^*v^*)^{1/2}\psi(x^*x)^{1/2} \\ &\leq \psi(vaa^*v^*)^{1/2}\|\lambda\|^{1/2}\|x\|.\end{aligned}$$

The latter inequality is shown in a similar way. □

2.2. **A topology of semigroups of endomorphisms.** Let M be a factor of type III. Let $\mathrm{End}(M)_0$ be the set of all finite index endomorphisms ρ of M. Let $d(\rho)$ be the square root of the minimal index of $M \supset \rho(M)$ and E_ρ be the minimal expectation from M to $\rho(M)$. Set $\phi_\rho := \rho^{-1} \circ E_\rho$. In Masuda–Tomatsu [15], a topology of $\mathrm{End}(M)_0$ is introduced in the following way. We have

$$\rho_i \to \rho$$

if, by definition, $\|\psi \circ \phi_{\rho_i} - \psi \circ \phi_\rho\| \to 0$ for any $\psi \in M_*$.

2.3. **Canonical extension of endomorphisms.** Let φ be a normal faithful semifinite weight of M and σ^φ be the group of modular automorphisms of φ. In Izumi [6], an extension $\tilde{\rho}$ of $\rho \in \mathrm{End}(M)_0$ on the continuous core $\tilde{M} := M \rtimes_{\sigma^\varphi} \mathbf{R}$ is introduced in the following way. We have

$$\tilde{\rho}(x\lambda_t^{\sigma^\varphi}) = d(\rho)^{it}\rho(x)[D\varphi \circ \phi_\rho : D\varphi]_t \lambda_t^{\sigma^\phi}$$

for $t \in \mathbf{R}$, $x \in M$, where $[D\varphi \circ \phi_\rho : D\varphi]_t$ is the Connes cocycle between $\varphi \circ \phi_\rho$ and φ. This extension does not depend on the choice of φ under a specific identification (See Theorem 2.4 of Izumi [6]). The extension $\tilde{\rho}$ is said to be the canonical extension of ρ.

In Lemma 3.5 of Masuda–Tomatsu [15], it is shown that there exists a left inverse $\phi_{\tilde{\rho}}$ of $\tilde{\rho}$ satisfying

$$\phi_{\tilde{\rho}}(x\lambda_t^\varphi) = d(\rho)^{-it}\phi_\rho(x[D\phi : D\phi \circ \phi_\rho]_t)\lambda_t^\varphi$$

for $x \in M$, $t \in \mathbf{R}$.

3. The main theorem

The main theorem of this paper is the following.

Theorem 2. *Let ρ , σ be endomorphisms of an AFD factor M of type* III *with $d(\rho), d(\sigma) < \infty$. Then the following two conditions are equivalent.*

(1) *We have $\phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho)/d(\sigma))}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}$.*
(2) *There exists a sequence $\{u_n\}$ of unitaries of M with $\mathrm{Ad}u_n \circ \rho \to \sigma$ as $n \to \infty$.*

As a Corollary, we have the following result.

Corollary 3. *Let M be an AFD factor and R_0 be the AFD factor of type* II_1. *Take endomorphisms ρ_1, $\rho_2 \in \mathrm{End}(M)_0$ and σ_1, $\sigma_2 \in^m athrmEnd(R_0)_0$. Then the following two conditions are equivalent.*

(1) *There exists a sequence of unitaries $\{u_n\}$ of $M \otimes R_0$ with $\mathrm{Ad}u_n \circ (\rho_1 \otimes \sigma_1) \to \rho_2 \otimes \sigma_2$ as $n \to \infty$.*

(2) *There exists a sequence of unitaries $\{v_n\}$ of M with $\mathrm{Ad}v_n \circ \rho_1 \to \rho_2$ as $n \to \infty$.*

Proof. Since σ_1 and σ_2 are approximately inner, we may assume that $\sigma_1 = \sigma_2 = \mathrm{id}_{R_0}$. By the identification $\mathcal{Z}((M \otimes R_0) \rtimes_{\sigma^\varphi \otimes \mathrm{id}_{R_0}} \mathbf{R}) \cong \mathcal{Z}((M \rtimes_{\sigma^\varphi} \mathbf{R}) \otimes R_0) \cong \mathcal{Z}(M \rtimes_{\sigma^\varphi} \mathbf{R})$ by

$$(x \otimes y)\lambda_t^{\sigma^\varphi \otimes \mathrm{id}_{R_0}} \mapsto (x\lambda_t^{\sigma^\varphi}) \otimes y,$$

we have $\phi_{\rho_i \otimes \mathrm{id}_{R_0}} = \phi_{\rho_i}$ for $i = 1, 2$. We also have $(\rho_i \otimes \mathrm{id}_{R_0}) = d(\rho_i)$. Hence by Theorem 2, conditions (1) and (2) are equivalent. $\square$

Note that this corollary would be quite difficult to show without Theorem 2 (See also Section 3 of Connes [2]).

Theorem 2 should also be useful for classifying actions of compact groups on the AFD factors of type III. Popa–Wassermann [21] and Masuda–Tomatsu [17] showed that any compact group has only one minimal action on the AFD factor of type II_1, up to conjugacy. One of the next problems is to classify actions of compact groups on the AFD factors of type III. In Masuda–Tomatsus [16] and [18], they are trying to solve this problem, and some partial answers to this problem are obtained (Theorems A, B of [16] and Theorem 2.4 of [18]). However, still the problem has not been solved completely. In Masuda–Tomatsu [16], a conjecture about this classification problem is proposed (Conjecture 8.2). Our main theorem implies that if two actions of discrete Kac algebras on the AFD factors of type III have the same invariants, the difference of these two actions is approximately inner (See Problem 8.3 and the preceding argument to that problem of Masuda–Tomatsu [16]). In order to classify group actions, whether the difference of two actions is approximately inner or not is very important. Kawahigashi–Sutherland–Takesaki [11] and Masuda–Tomatsu [15] characterize the approximate innerness of endomorphisms under such a motivation. Theorem 2 is a generalization of their results.

In the following, we will show Theorem 2. Implication (2)$\Rightarrow$ (1) is shown easily by using known results.

Proof of implication (2) $\Rightarrow$ (1) *of Theorem 2.* This is shown by the same argument as that of the proof of implication (1) $\Rightarrow$ (2) of Theorem 3.15 of [15]. Assume that we have $\mathrm{Ad}u_n \circ \rho \to \sigma$ as $n \to \infty$. Then by the continuity of normalized canonical extension (Theorem 3.8 of Masuda–Tomatsu [15]), we have

$$\phi_{\tilde{\rho}} \circ \theta_{-\log d(\rho)} \circ \mathrm{Ad}u_n^*(x) \to \phi_{\tilde{\sigma}} \circ \theta_{-\log d(\sigma)}(x)$$

in the strong* topology for any $x \in \tilde{M}$. Hence we have

$$\phi_{\tilde{\rho}} \circ \theta_{-\log(d(\rho)/d(\sigma))}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}.$$

$\square$

In the following, we will show the opposite implication. Our strategy is to reduce the problem to that of endomorphisms on semifinite von Neumann algebras. In order to achieve this, in Kawahigashi–Sutherland–Takesaki [11] and Masuda–Tomatsu [15], they have used discrete decomposition theorems (See Connes [3]). However, in our situation, the centers of the images of caonical extensions may not coincide with that of $\tilde{M}$. This makes the problem difficult. It seems that Corollary 4.4 of Izumi [6] means that it is difficult to show Theorem 2 by the same strategy as those in them. Instead, we will use continuous decomposition. We also note that our method gives a proof of Theorem (1) of Kawahigashi–Sutherland–Takesaki [11] which does not depend on the types of the AFD factors.

4. Approximation on the continuous core

In order to prove implication (1) $\Rightarrow$ (2) of Theorem 2, we need to prepare some lemmas. We first show the implication when $\phi_{\tilde{\rho}} = \phi_{\tilde{\sigma}}$. Until the end of the proof of Lemma 23, we always assume that $d(\rho) = d(\sigma)$ and $\phi_{\tilde{\rho}} = \phi_{\tilde{\sigma}}$. Choose a dominant weight φ of M (For the definition of dominant weights, see Definition II.1.2. and Theorem II.1.3. of Connes–Takesaki [4]). Then by Lemma 2.3 (3) of Izumi [6], it is possible to choose unitaries u and v of M so that $(\varphi, \mathrm{Ad}u \circ \rho)$ and $(\varphi, \mathrm{Ad}v \circ \sigma)$ are invariant pairs (See Definition 2.2 of Izumi [6]). More precisely, we have

$$\varphi \circ \mathrm{Ad}u \circ \rho = d(\rho)\varphi, \ \ \varphi \circ E_{\mathrm{Ad}u\circ\rho} = \varphi,$$

$$\varphi \circ \mathrm{Ad}v \circ \sigma = d(\sigma)\varphi, \ \ \varphi \circ E_{\mathrm{Ad}v\circ\sigma} = \varphi.$$

By replacing ρ by $\mathrm{Ad}u\circ\rho$ and σ by $\mathrm{Ad}v\circ\sigma$ respectively, we may assume that (φ, ρ) and (φ, σ) are invariant pairs. In the rest of this paper, we identify $\tilde{M}$ with $M \rtimes_{\sigma^\varphi} \mathbf{R}$. Let h be a positive self-adjoint operator affiliated to $\tilde{M}$ satisfying $h^{-it} = \lambda_t^\varphi$. Let τ be a trace of $\tilde{M}$ defined by $\hat{\varphi}(h\cdot)$.

Lemma 4. *For $\rho \in \mathrm{End}(M)_0$, we have $\phi_{\tilde{\rho}} = \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}$, where $E_{\tilde{\rho}}$ is the conditional expectation with respect to τ.*

Proof. For $x \in M$ and $t \in \mathbf{R}$, we have

$$\begin{aligned}\tilde{\rho} \circ \phi_{\tilde{\rho}}(x\lambda_t^\varphi) &= \tilde{\rho}(d(\rho)^{-it}\phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t)\lambda_t^\varphi) \\ &= d(\rho)^{it} d(\rho)^{-it}\rho(\phi_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t))[D\varphi \circ \phi_\rho : D\varphi]_t\lambda_t^\varphi \\ &= E_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t)[D\varphi \circ \phi_\rho : D\varphi]_t\lambda_t^\varphi\end{aligned}$$

Since (φ, ρ) is an invariant pair, we have

$$[D\varphi \circ \phi_\rho : D\phi]_t = d(\rho)^{-it}.$$

Hence we have

$$E_\rho(x[D\varphi : D\varphi \circ \phi_\rho]_t)[D\varphi \circ \circ\phi_\rho : D\varphi]_t\lambda_t^\varphi = E_\rho(x)d(\rho)^{it}d(\rho)^{-it}\lambda_t^\varphi = E_\rho(x)\lambda_t^\varphi.$$

Hence by an argument of p.226 of Longo [13], it is shown that $\tilde{\rho}\circ\phi_{\tilde{\rho}}$ is the expectation with respect to τ. □

Lemma 5. *For $\rho \in \mathrm{End}(M)_0$, we have $\tau \circ \phi_{\tilde{\rho}} = d(\rho)^{-1}\tau$.*

Proof. By Lemma 4, we have $\phi_{\tilde{\rho}} = \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}$. On the other hand, by Proposition 2.5 (4) of Izumi [6], we have $\tau \circ \tilde{\rho} = d(\rho)\tau$. Hence we have

$$\begin{aligned}\tau \circ \phi_{\tilde{\rho}} &= d(\rho)^{-1}\tau \circ \tilde{\rho} \circ \phi_{\tilde{\rho}} \\ &= d(\rho)^{-1}\tau \circ \tilde{\rho} \circ \tilde{\rho}^{-1} \circ E_{\tilde{\rho}} \\ &= d(\rho)^{-1}\tau \circ E_{\tilde{\rho}} \\ &= d(\rho)^{-1}\tau.\end{aligned}$$

□

In the following, we identify $\mathcal{Z}(\tilde{M})$ with $L^\infty(X, \mu)$. Let

$$\tau = \int_X^\oplus \tau_x \ d\mu(x)$$

be the direct integral decomposition of τ.

Lemma 6. *Let ρ, σ be elements of* $\mathrm{End}(M)_0$. *Assume that $\phi_{\tilde{\rho}}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}$ and $d(\rho) = d(\sigma)$. For $a \in \tilde{M}_+$ with $\tau(a) < \infty$, set*

$$b := \tilde{\rho}(a) = \int_X^{\oplus} b_x \ d\mu(x),$$

$$c := \tilde{\sigma}(a) = \int_X^{\oplus} c_x \ d\mu(x).$$

Then we have

$$\tau_x(b_x) = \tau_x(c_x)$$

for almost every $x \in X$.

Proof. Take an arbitrary positive element z of $\mathcal{Z}(\tilde{M})_+$. Then we have

$$\begin{aligned}\tau(bz) &= \int_X \tau_x(b_x z_x) \ d\mu(x) \\ &= \int_X \tau_x(b_x) z_x \ d\mu(x).\end{aligned}$$

Similarly, we have

$$\tau(cz) = \int_X \tau_x(c_x) z_x \ d\mu(x).$$

On the other hand, by Lemma 5, we have

$$\begin{aligned}\tau(bz) &= d(\rho)\tau \circ \phi_{\tilde{\rho}}(bz) \\ &= d(\rho)\tau \circ \phi_{\tilde{\rho}}(\tilde{\rho}(a)z) \\ &= d(\rho)\tau \circ \tilde{\rho}^{-1} \circ E_{\tilde{\rho}}(\tilde{\rho}(a)z) \\ &= d(\rho)\tau \circ \tilde{\rho}^{-1}(\tilde{\rho}(a)E_{\tilde{\rho}}(z)) \\ &= d(\rho)\tau(a\phi_{\tilde{\rho}}(z)).\end{aligned}$$

Since we assume $d(\rho) = d(\sigma)$ and $\phi_{\tilde{\rho}}|_{\mathcal{Z}(\tilde{M})} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(\tilde{M})}$, the last number of the above equality is $d(\sigma)\tau(a\phi_{\tilde{\sigma}}(z))$, which is shown to be $\tau(cz)$ in a similar way. Hence we have

$$\int_X \tau_x(b_x) z_x \ d\mu(x) = \int_X \tau_x(c_x) z_x \ d\mu(x).$$

Since the maps $x \mapsto \tau_x(b_x)$ and $x \mapsto \tau_x(c_x)$ are integrable functions and $z \in L^\infty(X, \mu) = L^1(X, \mu)^*$ is arbitrary, we have $\tau_x(b_x) = \tau_x(c_x)$ for almost every $x \in X$. □

Note that we have never used the assumption that M is approximately finite up to this point. However, in order to show the following lemma, we need to assume that M is approximately finite. Let

$$\tilde{M} = \int_X^{\oplus} \tilde{M}_x \ d\mu(x)$$

be the direct integral decomposition.

Lemma 7. *Let M be an AFD factor of type* III *and ρ, σ be as in Lemma 6. Then for almost every $x \in X$, there exist a factor B_x of type I_∞, a unitary u of $\tilde{M}_x$ and a sequence $\{u_n\}$ of unitaries of $\tilde{M}_x$ with the following properties.*

(1) *The relative commutant $B_x' \cap \tilde{M}_x$ is finite.*

(2) *There exists a sequence of unitaries* $\{v_n\}$ *of* $B'_x \cap \tilde{M}_x$ *with* $u_n = (v_n \otimes 1)u$, *where we identify* $\tilde{M}_x$ *with* $(B'_x \cap \tilde{M}_x) \otimes B_x$.

(3) *For almost every* $x \in X$ *and for any* $a \in \tilde{M}$, *we have* $\mathrm{Ad}u_n((\tilde{\rho}(a))_x) \to (\tilde{\sigma}(a))_x$ *in the strong* * *topology.*

(4) *We have* $B_x \subset u(\tilde{\rho}(\tilde{M}))_x u^* \cap (\tilde{\sigma}(\tilde{M}))_x$.

Proof. Let $B_0 \subset \tilde{\rho}(\tilde{M})$ be a factor of type I_∞ with $Q := \tilde{\rho}(\tilde{M}) \cap B'_0$ finite. Let $\{f^0_{ij}\}$ be a matrix unit generating B_0. We may assume that $\tau(f^0_{ii}) < \infty$. Then since $(\tau \circ E_{\tilde{\rho}})_x((f^0_{11})_x) < \infty$ for almost every $x \in X$, $P := \tilde{M} \cap B'_0$ is also finite. Then by Lemma 6, there exists a partial isometry v of $\tilde{M}$ with $v^*v = \tilde{\rho}(f^0_{11})$, $vv^* = \tilde{\sigma}(f^0_{11})$. Set

$$u := \sum_{j=1}^{\infty} \tilde{\sigma}(f^0_{j1}) v \tilde{\rho}(f^0_{1j}).$$

Then u is a unitary of $\tilde{M}$ with $u\tilde{\sigma}(f^0_{ij})u^* = \tilde{\rho}(f^0_{ij})$. Set

$$B := \tilde{\sigma}(B_0) (= u\tilde{\rho}(B_0)u^*),$$

$$f_{ij} := \tilde{\sigma}(f^0_{ij}) (= u\tilde{\rho}(f^0_{ij})u^*).$$

By replacing $\tilde{\rho}$ by $\mathrm{Ad}u \circ \tilde{\rho}$, we may assume that $\tilde{\rho}(f_{ij}) = \tilde{\sigma}(f_{ij})$. In the following, we identify $\tilde{M}$ with $P \otimes B$ and P with $R \otimes \mathcal{Z}(\tilde{M})$, where R is the AFD factor of type II_1. By the approximate finiteness of R and $\mathcal{Z}(\tilde{M})$, there exists a sequence $\{\{e^n_{ij} \otimes a^n_k\}_{i,j,k}\}_{n=1}^{\infty}$ of systems of partial isometries of P with the following properties.

(1) For each n, the system $\{e^n_{ij}\}_{i,j}$ is a matrix unit of R.
(2) For each n, the system $\{a^n_k\}_k$ is a partition of unity in $\mathcal{Z}(\tilde{M})$.
(3) For each n, $\{e^{n+1}_{ij}\}_{i,j}$ is a refinement of $\{e^n_{ij}\}_{i,j}$.
(4) For each n, $\{a^{n+1}_k\}_k$ is a refinement of $\{a^n_k\}_k$.
(5) We have $\bigvee_{n=1}^{\infty} \{e^n_{ij} \otimes a^n_k\}''_{i,j,k} = P$.

Fix a natural number n. Then by Lemma 6, we have

$$\tau_x((\tilde{\rho}(e^n_{11} \otimes a^n_k \otimes f_{11}))_x) = \tau_x((\tilde{\sigma}(e^n_{11} \otimes a^n_k \otimes f_{11}))_x)$$

for almost every $x \in X$. Hence for almost every $x \in X$, there exists a partial isometry v^n_k of $P_x = (\tilde{\rho}(f_{11})\tilde{M}\tilde{\rho}(f_{11}))_x$ with

$$v^{n*}_k v^n_k = \tilde{\rho}(e^n_{11} \otimes a^n_k \otimes f_{11})_x, \ v^n_k v^{n*}_k = \tilde{\sigma}(e^n_{11} \otimes a^n_k \otimes f_{11})_x.$$

Set

$$v_n := \sum_{k,j} \tilde{\sigma}(e_{j1} \otimes a^n_k \otimes f_{11})_x v^n_k \tilde{\rho}(e_{1j} \otimes a^n_k \otimes f_{11})_x.$$

Then v_n is a unitary of $\tilde{\rho}(f_{11})_x \tilde{M}_x \tilde{\rho}(f_{11})_x$ with

$$v_n \tilde{\rho}(e^n_{ij} \otimes a^n_k \otimes f_{11})_x v^*_n = \tilde{\sigma}(e^n_{ij} \otimes a^n_k \otimes f_{11})_x.$$

Hence for almost every $x \in X$, there exists a sequence $\{v_n\}$ of unitaries of P_x with

$$\mathrm{Ad}(v_n \otimes 1)(\tilde{\rho}(a)_x) \to \tilde{\sigma}(a)_x$$

for any $a \in \tilde{M}$. □

Lemma 8. *Let M, ρ and σ be as in Lemma 7. Then there exist a unital subfactor B of $\tilde{M}$, a unitary u of $\tilde{M}$ and a sequence $\{u_n\}$ of unitaries of $\tilde{M}$ with the following properties.*

(1) *The factor B is of type I_∞.*
(2) *The relative commutant $B' \cap \tilde{M}$ is finite.*
(3) *There exists a sequence of unitaries $\{v_n\}$ of $B' \cap \tilde{M}$ with $u_n = (v_n \otimes 1)u$, where we identify $\tilde{M}$ with $(B' \cap \tilde{M}) \otimes B$.*
(4) *For any $a \in \tilde{M}$, we have $\mathrm{Ad}u_n \circ \tilde{\rho}(a) \to \tilde{\sigma}(a)$ in the strong * topology.*
(5) *We have $B \subset u\tilde{\rho}(\tilde{M})u^* \cap \tilde{\sigma}(\tilde{M})$.*

Proof. This is shown by "directly integrating" the above lemma. □

The conclusion of Lemma 8 means that $\mathrm{Ad}u_n \circ \tilde{\rho}$ converges to $\tilde{\sigma}$ point *strongly. However, this convergence is slightly weaker than the topology we consider. We need to fill this gap. In order to achieve this, the following criterion is very useful.

Lemma 9. (Lemma 3.8 of Masuda–Tomatsu [17]). *Let ρ and ρ_n, $n \in \mathbf{N}$ be be endomorphisms of a von Neumann algebra N with left inverses Φ and Φ_n, $n \in \mathbf{N}$, respectively. Fix a normal faithful state ϕ of N. Then the following two conditions are equivalent.*
(1) *We have $\lim_{n\to\infty} \|\psi \circ \Phi_n - \psi \circ \Phi\| = 0$ for all $\psi \in N_*$.*
(2) *We have $\lim_{n\to\infty} \|\phi \circ \Phi_n - \phi \circ \Phi\| = 0$ and $\lim_{n\to\infty} \rho_n(a) = \rho(a)$ for all $a \in N$.*

Hence what we need to do is to find a normal faithful state of $\tilde{M}$ satisfying condition (2) of Lemma 9.

Lemma 10. *Let M, ρ, σ be as in Lemma 7. Then there exists a sequence of unitaries u_n of $\tilde{M}$ with $\mathrm{Ad}u_n \circ \tilde{\rho} \to \tilde{\sigma}$.*

Proof. Take a subfactor B of $\tilde{M}$, a unitary u of $\tilde{M}$ and a sequence $\{v_n\}$ of unitaries of $\tilde{M}$ as in Lemma 8. By condition (5) in Lemma 8, we have $u^*Bu \subset \tilde{\rho}(\tilde{M})$. Set

$$F := \tilde{\rho}^{-1}(u^*Bu).$$

Then we have

$$\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*(B) = F,$$

$$\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*(B' \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M})) = F' \cap \tilde{M}.$$

We also have

$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*|_B = \mathrm{id}_B,$$

$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(B' \cap \tilde{M}) = B' \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M}).$$

Let $\{f_{ij}\}$ be a matrix unit generating B. Set

$$\overline{\tau}(a) := \tau(a\tilde{\rho}^{-1}(u^* f_{11} u))$$

for $a \in F' \cap \tilde{M}$, which is a faithful normal finite trace of $F' \cap \tilde{M}$. Let φ be a normal faithful state of F. Let $\Psi_F : \tilde{M} \to (F' \cap \tilde{M}) \otimes F$ is the natural identification map.

Then by the above observation, for $a \in B' \cap \tilde{M}$ and i, j, we have

$$\begin{aligned}
&(\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*(af_{ij}) \\
&= (\overline{\tau} \otimes \varphi) \circ \Psi_F \circ (\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*) \circ (\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(af_{ij}) \\
&= (\overline{\tau} \otimes \varphi) \circ \Psi_F \circ (\tilde{\rho}^{-1} \circ \mathrm{Ad}u^*)((\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*|_{B' \cap \tilde{M}})(a)f_{ij}) \\
&= (\overline{\tau} \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(a)(\varphi \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*)(f_{ij}).
\end{aligned}$$

Since $B \subset \tilde{\sigma}(\tilde{M}) \cap \mathrm{Ad}u \circ \tilde{\rho}(\tilde{M})$, we have

$$E_{\tilde{\sigma}}(af_{ij}) = E_{\tilde{\sigma}}(a)f_{ij},$$
$$\mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(af_{ij}) = \mathrm{Ad}u \circ E_{\tilde{\rho}} \circ \mathrm{Ad}u^*(a)f_{ij}$$

for $a \in B' \cap \tilde{M}$. Notice that $\tilde{\sigma}^{-1}(f_{ij}) = \tilde{\rho}^{-1}(u^*f_{ij}u)$ by condition (3) of Lemma 8. Then for any $a \in B' \cap \tilde{M}$, we have

$$\begin{aligned}
&(\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}u^*(v_n^* \otimes 1)(af_{ij}) \\
&= (\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\rho}}((u^*(v_n^* a v_n)u)(u^* f_{ij} u)) \\
&= \overline{\tau} \circ \phi_{\tilde{\rho}}(u^*(v_n^* a v_n)u)\varphi(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= \tau(\phi_{\tilde{\rho}}(u^*(v_n^* a v_n)u)\tilde{\rho}^{-1}(u^* f_{11} u))\varphi(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= \tau \circ \phi_{\tilde{\rho}}(u^*(v_n^* a v_n) f_{11} u)\varphi(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= d(\rho)\tau(u^*(v_n^* a v_n) f_{11} u)\varphi(\tilde{\rho}^{-1}(u^* f_{ij} u)) \\
&= d(\sigma)\tau(af_{11})\varphi(\tilde{\sigma}^{-1}(f_{ij})) \\
&= \tau(\phi_{\tilde{\sigma}}(a)\tilde{\sigma}^{-1}(f_{11}))\varphi(\tilde{\sigma}^{-1}(f_{ij})) \\
&= \tau(\phi_{\tilde{\sigma}}(a)\tilde{\rho}^{-1}(u^* f_{11} u))\varphi(\tilde{\sigma}^{-1}(f_{ij})) \\
&= (\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\sigma}}(af_{ij}).
\end{aligned}$$

Hence we have $(\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\rho}} \circ \mathrm{Ad}(u^*(v_n \otimes 1)^*) = (\overline{\tau} \otimes \varphi) \circ \Psi_F \circ \phi_{\tilde{\sigma}}$ for any n. Hence by Lemma 8 and Lemma 9, we have $\mathrm{Ad}((v_n \otimes 1)u) \circ \tilde{\rho} \to \tilde{\sigma}$. $\square$

5. Averaging by the trace-scaling action

In this section, we always assume that M is an AFD factor of type III. Let φ be a dominant weight of M and $\rho, \sigma \in \mathrm{End}(M)_0$ be finite index endomorphisms with (φ, ρ) and (φ, σ) invariant pairs. Set

$$\tilde{M} := M \rtimes_{\sigma^\varphi} \mathbf{R}.$$

Let ψ_0 be a normal faithful state of $\tilde{M}$ and $\{\psi_i\}_{i=1}^\infty$ be a norm dense sequence of the unit ball of $\tilde{M}_*$. Let θ be the dual action on $\tilde{M}$ of σ^φ. We will replace the sequence $\{u_n\}$ chosen in the previous section so that it is almost invariant by θ. In order to achieve this, we use a property of θ which is said to be the Rohlin property. In order to explain this property, we first need to explain related things. Let ω be an ultrafilter of $\mathbf{N}$. A sequence $\{[-1,1] \ni t \mapsto x_{n,t} \in \tilde{M}\}_{n=1}^\infty$ of maps from $[-1,1]$ to $\tilde{M}$ is said to be ω-equicontinuous if for any $\epsilon > 0$, there exist an element $U \subset \mathbf{N}$ of ω and $\delta > 0$ with $\|x_{n,t} - x_{n,s}\| < \epsilon$ for any $s, t \in [-1,1]$ with $|s-t| < \delta$, $n \in U$. Set

$$\mathcal{C} := \{(x_n) \in l^\infty(\tilde{M}) \mid \|x_n\psi - \psi x_n\| \to 0 \text{ for any } \psi \in \tilde{M}.\},$$
$$\mathcal{C}_{\theta,\omega} := \{(x_n) \in \mathcal{C}_\omega \mid \text{the maps } \{t \mapsto \theta_t(x_n)\}_{n=1}^\infty \text{ are } \omega \text{ equicontinuous.}\},$$
$$\mathcal{I}_\omega := \{(x_n) \in l^\infty(\tilde{M}) \mid x_n \to 0 \text{ in the } *\text{strong topology.}\}.$$

Then $\mathcal{I}_\omega$ is a (norm) closed ideal of $\mathcal{C}_{\theta,\omega}$, and the quotient $\tilde{M}_{\theta,\omega} := \mathcal{C}_{\theta,\omega}/\mathcal{I}_\omega$ is a von Neumann algebra. As mentioned in Masuda–Tomtasu [19], the action θ has the Rohlin property, that is, for any $R > 0$, there exists a unitary v of $\tilde{M}_{\theta,\omega}$ with

$$\theta_t(v) = e^{-iRt}v$$

for any $t \in \mathbf{R}$ (See Section 4 of Masuda–Tomatsu [19]). Choose arbitrary numbers $r > 0$ and $0 < \epsilon < 1$. Then since M is of type III, there exists a real number R which is not of the discrete spectrum of $\theta|_{\mathcal{Z}(\tilde{M})}$ and which satisfies $r/R < \epsilon^2$. Then as shown in Theorem 5.2 of Masuda–Tomatsu [19], there exists a normal injective *-homomorphism Θ from $\tilde{M} \otimes L^\infty([-R,R])$ to $\tilde{M}^\omega$ satisfying $x \otimes f \mapsto xf(v)$ for any $x \in \tilde{M}$, $f \in L^\infty([-R,R])$. For each $t \in \mathbf{R}$, set

$$\gamma_t : L^\infty([-R,R]) \ni f \mapsto f(\cdot - t) \in L^\infty([-R,R]),$$

where we identify $[-R,R]$ with $\mathbf{R}/2R\mathbf{Z}$ as measured spaces. Then the *-homomorphisms Θ and γ_t satisfy

$$\Theta \circ (\theta_t \otimes \gamma_t) = \theta_t \circ \Theta$$

(See Theorem 5.2 of Masuda–Tomatsu [19]).

Lemma 11. *For $\psi \in \tilde{M}_*$ and $x \otimes f \in \tilde{M} \otimes L^\infty([-R,R])$, we have*

$$\psi^\omega \circ \Theta = \psi \otimes \tau_{L^\infty},$$

where τ_{L^∞} is the trace coming from the normalized Haar measure of $L^\infty([-R,R])$.

Proof. Let $\{v_n\}$ be a representing sequence of v. Then we have

$$\begin{aligned} \psi^\omega \circ \Theta(x \otimes f) &= \psi^\omega(xf(v)) \\ &= \lim_{n\to\omega} \psi(xf(v_n)) \\ &= \psi(x) \lim_{n\to\omega} f(v_n) \\ &= \psi(x)\tau_{L^\infty}(f) \\ &= (\psi \otimes \tau_{L^\infty})(x \otimes f). \end{aligned}$$

□

Since the maps

$$[-R,R] \ni t \mapsto \psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t \in (\tilde{M})_*,$$

$$[-R,R] \ni t \mapsto \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t \in (\tilde{M})_*$$

are norm continuous, the union of their images

$$\{\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t \mid t \in [-R,R]\} \cup \{\psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t \mid t \in [-R,R]\}$$

is compact. Hence there exists a finite set $-R = t_0 < \cdots < t_J = R$ of $[-R,R]$ such that

$$\|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_{t_j} - \psi_i \circ \phi_{\tilde{\rho}} \circ \theta_t\| < \epsilon,$$

$$\|\psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j} - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t\| < \epsilon$$

for any $i = 1, \cdots, n$, $j = 0, \cdots, J-1$ and $t \in [t_j, t_{j+1}]$. We may assume that $t_j = 0$ for some j. Then by Lemma 10, there exists a unitary u of $\tilde{M}$ with

$$\|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_{t_j} \circ \mathrm{Ad}u - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j}\| < \epsilon$$

for any $j = 0, \cdots, J-1$, $i = 1, \cdots, n$ (Notice that we used the fact that we have $\phi_{\tilde{\rho}} \circ \theta_{t_j} = \theta_{t_j} \circ \phi_{\tilde{\rho}}$ and that we have $\phi_{\tilde{\sigma}} \circ \theta_{t_j} = \theta_{t_j} \circ \phi_{\tilde{\sigma}}$ for any $j = 0, \cdots, J-1$). Set

$$U : [-R, R] \ni t \mapsto \theta_t(u) \in \tilde{M},$$

which is a unitary of $\tilde{M} \otimes L^\infty([-R, R])$.

Lemma 12. *We have*

$$\|(\psi_i \circ \phi_{\tilde{\rho}})^\omega \circ \mathrm{Ad}\Theta(U)|_{\mathrm{Im}\Theta} - (\psi_i \circ \phi_{\tilde{\sigma}})^\omega|_{\mathrm{Im}\Theta}\| < 3\epsilon.$$

Proof. Let m be the normalized Haar measure of $[-R, R]$. By Lemma 11, we have

$$\begin{aligned}
&\|(\psi_i \circ \phi_{\tilde{\rho}})^\omega \circ \mathrm{Ad}\Theta(U)|_{\mathrm{Im}\Theta} - (\psi_i \circ \phi_{\tilde{\sigma}})^\omega|_{\mathrm{Im}\Theta}\| \\
&= \|((\psi_i \circ \phi_{\tilde{\rho}}) \otimes \tau_{L^\infty}) \circ \mathrm{Ad}U - (\psi_i \circ \phi_{\tilde{\sigma}}) \otimes \tau_{L^\infty}\| \\
&= \int_{[-R,R]} \|(\psi_i \circ \phi_{\tilde{\rho}}) \circ \mathrm{Ad}\theta_t(u) - \psi_i \circ \phi_{\tilde{\sigma}}\| \ dm(t) \\
&= \int_{[-R,R]} \|(\psi_i \circ \phi_{\tilde{\rho}}) \circ \theta_t \circ \mathrm{Ad}u - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t\| \ dm(t) \\
&\leq \sum_{j=0}^{J-1} \int_{[t_j, t_{j+1}]} (\|(\psi_i \circ \phi_{\tilde{\rho}}) \circ \theta_t - (\psi_i \circ \phi_{\tilde{\rho}}) \circ \theta_{t_j}\| \\
&+ \|\psi_i \circ \phi_{\tilde{\rho}} \circ \theta_{t_j} \circ \mathrm{Ad}u - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j}\| \\
&+ \|\psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_{t_j} - \psi_i \circ \phi_{\tilde{\sigma}} \circ \theta_t\|) \ dm(t) \\
&\leq \sum_{j=0}^{J-1} \int_{[t_j, t_{j+1}]} (\epsilon + \epsilon + \epsilon) \ dm(t) \\
&= 3\epsilon.
\end{aligned}$$

□

Lemma 13. *We have*

$$\|\theta_s(\Theta(U)) - \Theta(U)\|^\sharp_{\psi_0^\omega} < 2\epsilon$$

for $|s| \leq r$.

Proof. Notice that we have

$$(\theta_s \otimes \gamma_s)(U) : t \mapsto \theta_s(U_{t-s}),$$

where U_t denotes the evaluation of the function U at the point t. Hence by the definition of U, we have

$$(\theta_s \otimes \gamma_s)(U)_t = \theta_t(u)$$

for any $t \in [-R+r, R-r]$, where the left hand side is the evaluation of the function $(\theta_s \otimes \gamma_s)(U)$ at the point t. Hence by Lemma 11, we have

$$\begin{aligned}
&\|\theta_s(\Theta(U)) - \Theta(U)\|^\sharp_{\psi_0^\omega} \\
&= \|(\theta_s \otimes \gamma_s)(U) - U\|^\sharp_{\psi_0 \otimes \tau_{L^\infty}} \\
&= (\int_{[-R,R]} (\|((\theta_s \otimes \gamma_s)(U))_t - U_t\|^\sharp_{\psi_0})^2 \ dm(t))^{1/2} \\
&\leq (\int_{[-R,-R+r]\cup[R-r,R]} 4 \ dm(t))^{1/2} \\
&\leq (4\epsilon^2)^{1/2} \\
&= 2\epsilon.
\end{aligned}$$

□

Let

$$\psi_i \circ \phi_{\tilde{\sigma}} = |\psi_i \circ \phi_{\tilde{\sigma}}| v_i$$

be the polar decompositions of $\psi_i \circ \phi_{\tilde{\sigma}}$ for $i = 1, \cdots, n$.

Lemma 14. *There exists a finite subset $-R = s_0 < \cdots < s_K = R$ of $[-R, R]$ with the following properties.*

(1) *We have*

$$\|(U - \sum_{k=0}^{K-1} \theta_{s_k}(u) e_k) v_i^*\|^\sharp_{|\psi_i \circ \phi_{\tilde{\sigma}}| \otimes \tau_{L^\infty}} < \epsilon$$

for any $i = 1, \cdots, n$, where $e_k := \chi_{[s_k, s_{k+1}]} \in L^\infty([-R, R])$.

(2) *We have*

$$\|U - \sum_{k=0}^{K-1} \theta_{s_k}(u) e_k\|^\sharp_{|\psi_i \circ \phi_{\tilde{\rho}}| \otimes \tau_{L^\infty}} < \epsilon$$

for any $i = 1, \cdots, n$.

(3) *We have*

$$\|U - \sum_{k=0}^{K-1} \theta_{s_k}(u) e_k\|^\sharp_{(\psi_0 \circ \theta_{t_j}) \otimes \tau_{L^\infty}} < \epsilon$$

for any $i = 1, \cdots, n$ and $j = 0, \cdots, J-1$.

Proof. Since the map $t \mapsto \theta_t(u)$ is continuous in the strong * topology, there exists a finite set $-R = s_0 < \cdots < s_K = R$ of $[-R, R]$ with

$$\|(\theta_t(u) - \theta_{s_k}(u)) v_i^*\|^\sharp_{|\psi_i \circ \phi_{\tilde{\sigma}}|} < \epsilon$$

for $i = 1, \cdots, n$, $k = 0, \cdots, K-1$ and $t \in [s_k, s_{k+1}]$,

$$\|\theta_t(u) - \theta_{s_k}(u)\|^\sharp_{|\psi_i \circ \phi_{\tilde{\rho}}|} < \epsilon$$

for $i = 1, \cdots, n$, $k = 0, \cdots, K-1$ and $t \in [s_k, s_{k+1}]$,

$$\|\theta_t(u) - \theta_{s_k}(u)\|^\sharp_{\psi_0 \circ \theta_{t_j}} < \epsilon$$

for $j = 0, \cdots, J-1$, $k = 0, \cdots, K-1$ and $t \in [s_k, s_{k+1}]$. Then we have

$$\begin{aligned}
&\|(U - \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k)v_i^*\|^\sharp_{|\psi_i \circ \phi_{\tilde{\sigma}}| \otimes \tau_{L^\infty}} \\
&= (\sum_{k=0}^{K-1} \int_{[s_k, s_{k+1})} (\|(\theta_t(u) - \theta_{s_k}(u))v_i^*\|^\sharp_{|\psi_i \circ \phi_{\tilde{\sigma}}|})^2 \ dm(t))^{1/2} \\
&< (\sum_{k=0}^{K-1} \int_{[s_k, s_{k+1})} \epsilon^2 \ dm(t))^{1/2} \\
&= \epsilon.
\end{aligned}$$

The other inequalities are shown in a similar way. □

Set

$$V := \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k.$$

Take a representing sequence $\{e_k^n\}_{n=1}^\infty$ of $\Theta(e_k)$ so that $\{e_k^n\}_{k=0}^{K-1}$ is a partition of unity in $\tilde{M}$ by projections for each n. Set

$$v_n := \sum_{k=0}^{K-1} \theta_{s_k}(u)e_k^n,$$

which is a unitary. The sequence $\{v_n\}_{n=1}^\infty$ represents the unitary $\Theta(V)$. Let $\{u_n\}_{n=1}^\infty$ be a representing sequence of $\Theta(U)$.

Lemma 15. *We have*

$$\lim_{n \to \omega} \|\theta_t(v_n) - v_n\|^\sharp_{\psi_0} < 6\sqrt{\epsilon}.$$

for $t \in [-r, r]$.

Proof. Note that we have

$$\begin{aligned}
&(\|\theta_t(a)\|^\sharp_{\psi_0})^2 \\
&= \frac{1}{2}\psi_0 \circ \theta_t(a^*a + aa^*) \\
&= \frac{1}{2}(\psi_0 \circ \theta_{t_j}(a^*a + aa^*)) - \frac{1}{2}((\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t)(a^*a + aa^*)) \\
&\leq (\|a\|^\sharp_{\psi_0 \circ \theta_{t_j}})^2 + \|a\|^2 \|\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t\|
\end{aligned}$$

for any $a \in \tilde{M}$. Hence for $t \in [t_j, t_{j+1}] \cap [-r, r]$, we have

$$\begin{aligned}
&\|\theta_t(v_n) - v_n\|^\sharp_{\psi_0} \\
&\leq \|\theta_t(v_n - u_n)\|^\sharp_{\psi_0} + \|\theta_t(u_n) - u_n\|^\sharp_{\psi_0} + \|u_n - v_n\|^\sharp_{\psi_0} \\
&\leq (4\|\psi_0 \circ \theta_{t_j} - \psi_0 \circ \theta_t\| + (\|v_n - u_n\|^\sharp_{\psi_0 \circ \theta_{t_j}})^2)^{1/2} \\
&+ \|\theta_t(u_n) - u_n\|^\sharp_{\psi_0} + \|u_n - v_n\|^\sharp_{\psi_0} \\
&< (4\epsilon + (\|v_n - u_n\|^\sharp_{\psi_0 \circ \theta_{t_j}})^2)^{1/2} \\
&+ \|\theta_t(u_n) - u_n\|^\sharp_{\psi_0} + \|u_n - v_n\|^\sharp_{\psi_0}.
\end{aligned}$$

Hence by Lemmas 13 and 14 (3), we have

$$\begin{aligned}
&\lim_{n\to\omega}\|\theta_t(v_n)-v_n\|_{\psi_0}^\sharp \\
&\leq (4\epsilon+(\|V-U\|^\sharp_{(\psi_0\circ\theta_{t_j})\otimes\tau_{L^\infty}})^2)^{1/2} \\
&+\|\theta_t(U)-U\|^\sharp_{\psi_0\otimes\tau_{L^\infty}}+\|U-V\|^\sharp_{\psi_0\otimes\tau_{L^\infty}} \\
&<(4\epsilon+\epsilon^2)^{1/2}+2\epsilon+\epsilon \\
&<6\sqrt{\epsilon}.
\end{aligned}$$

□

Lemma 16. *We have*

$$\lim_{n\to\omega}\|v_n^*\psi_i\circ\phi_{\tilde{\rho}}-\psi_i\circ\phi_{\tilde{\sigma}}v_n^*\|\leq 7\epsilon$$

for any $i=1,\cdots,n$.

Proof. Notice that we have

$$\begin{aligned}
&\|u_n^*(\psi_i\circ\phi_{\tilde{\rho}})-(\psi_i\circ\phi_{\tilde{\sigma}})u_n^*\| \\
&\|\Theta(U)^*(\psi_i\circ\phi_{\tilde{\rho}})^\omega|_M-(\psi_i\circ\phi_{\tilde{\sigma}})^\omega\Theta(U)^*|_M\| \\
&\leq\|(\psi_i\circ\phi_{\tilde{\rho}})^\omega\circ\mathrm{Ad}\Theta(U)|_{\mathrm{Im}\Theta}-(\psi_i\circ\phi_{\tilde{\sigma}})^\omega|_{\mathrm{Im}\Theta}\|.
\end{aligned}$$

Hence by Lemmas 12 and 14 (1) (2), we have

$$\begin{aligned}
&\lim_{n\to\omega}\|v_n^*\psi_i\circ\phi_{\tilde{\rho}}-\psi_i\circ\phi_{\tilde{\sigma}}v_n^*\| \\
&\leq\lim_{n\to\omega}(\|(v_n^*-u_n^*)\psi_i\circ\phi_{\tilde{\rho}}\| \\
&+\|u_n^*\psi_i\circ\phi_{\tilde{\rho}}-\psi_i\circ\phi_{\tilde{\sigma}}u_n^*\|+\|\psi_i\circ\phi_{\tilde{\sigma}}(u_n^*-v_n^*)\|) \\
&\leq\lim_{n\to\omega}(\|(v_n-u_n)^*\|_{|\psi_i\circ\phi_{\tilde{\rho}}|} \\
&+\|(\psi_i\circ\phi_{\tilde{\rho}})^\omega\circ\mathrm{Ad}\Theta(U)|_{\mathrm{Im}\Theta}-(\psi_i\circ\phi_{\tilde{\sigma}})^\omega|_{\mathrm{Im}\Theta}\| \\
&+\|(v_n-u_n)v_i^*\|_{|\psi_i\circ\phi_{\tilde{\sigma}}|}) \\
&=\|V-U\|_{|\psi_i\circ\phi_{\tilde{\rho}}|\otimes\tau_{L^\infty}} \\
&+\|(\psi_i\circ\phi_{\tilde{\rho}})^\omega\circ\mathrm{Ad}\Theta(U)|_{\mathrm{Im}\Theta}-(\psi_i\circ\phi_{\tilde{\sigma}})^\omega|_{\mathrm{Im}\Theta}\|+\|(V-U)v_i^*\|_{|\psi_i\circ\phi_{\tilde{\sigma}}|\otimes\tau_{L^\infty}} \\
&\leq\epsilon+3\epsilon+\epsilon \\
&=5\epsilon.
\end{aligned}$$

Note that in order to show the second inequality, we used Lemma 1. □

By Lemmas 15 and 16, we have the following proposition.

Proposition 17. *There exists a sequence* $\{v_n\}_{n=1}^\infty$ *of unitaries of* $\tilde{M}$ *with*

$$\lim_{n\to\infty}\|\theta_t(v_n)-v_n\|^\sharp_{\psi_0}=0,$$

$$\lim_{n\to\infty}\|v_n^*\psi_i\circ\phi_{\tilde{\rho}}-\psi_i\circ\phi_{\tilde{\sigma}}v_n^*\|=0$$

for any $i=1,2,\cdots$.

6. Approximation on $\tilde{M} \rtimes_\theta \mathbf{R}$.

Set
$$n_\tau := \{x \in \tilde{M} \mid \tau(x^* x) < \infty\}.$$

Lemma 18. *Let $L^2(\tilde{M})$ be the standard Hilbert space of $\tilde{M}$ and $\Lambda : n_\tau \to L^2(\tilde{M})$ be the canonical injection. For each $x \in n_\tau$, set $V_{\tilde{\rho}}(\Lambda(x)) := \sqrt{d(\rho)}^{-1}\Lambda(\tilde{\rho}(x))$. Then $V_{\tilde{\rho}}$ defines an isometry of $L^2(\tilde{M})$ satisfying*
$$V_{\tilde{\rho}}^* x V_{\tilde{\rho}} = \phi_{\tilde{\rho}}(x)$$
for any $x \in \tilde{M}$.

Proof. Take $x \in n_\tau$. Then by Lemma 2.5 (4) of Izumi [6], we have
$$\begin{aligned}\|V_{\tilde{\rho}}\Lambda(x)\|^2 &= d(\rho)^{-1}\tau(\tilde{\rho}(x^* x)) \\ &= \tau(x^* x) = \|\Lambda(x)\|^2.\end{aligned}$$
Hence $V_{\tilde{\rho}}$ defines an isometry of $L^2(\tilde{M})$. Next, we show the latter statement. We have $V_{\tilde{\rho}}^*(\Lambda(x)) = \sqrt{d(\rho)}\Lambda(\phi_{\tilde{\rho}}(x))$ because
$$\begin{aligned}\langle V_{\tilde{\rho}}^*\Lambda(x), \Lambda(y)\rangle &= \langle \Lambda(x), \sqrt{d(\rho)}^{-1}\Lambda(\tilde{\rho}(y))\rangle \\ &= \sqrt{d(\rho)}^{-1}\tau(\tilde{\rho}(y)^* x) \\ &= \sqrt{d(\rho)}\tau(y^*\phi_{\tilde{\rho}}(x)) \\ &= \langle \sqrt{d(\rho)}\Lambda(\phi_{\tilde{\rho}}(x)), \Lambda(y)\rangle\end{aligned}$$
for any $x, y \in n_\tau$. In order to show the third equality of the above, we used Lemma 5. Hence for any $x \in \tilde{M}$ and $y \in n_\tau$, we have
$$\begin{aligned}V_{\tilde{\rho}}^* x V_{\tilde{\rho}}\Lambda(y) &= \sqrt{d(\rho)}^{-1} V_{\tilde{\rho}}^*\Lambda(x\tilde{\rho}(y)) \\ &= \Lambda(\phi_{\tilde{\rho}}(x\tilde{\rho}(y))) \\ &= \phi_{\tilde{\rho}}(x)\Lambda(y).\end{aligned}$$
□

Let ρ be an endomorphism of a von Neumann algebra M. Then since its canonical extension $\tilde{\rho}$ satisfies $\tau \circ \tilde{\rho} = d(\rho)\tau$, the endomorphism $\tilde{\rho}$ extends to $\tilde{M} \rtimes_\theta \mathbf{R}$ by $\lambda_t^\theta \mapsto \lambda_t^\theta$ for any $t \in \mathbf{R}$. We denote this extension by $\tilde{\tilde{\rho}}$.

Lemma 19. *Let α and σ be finite index endomorphisms of a separable infinite factor M and φ be a dominant weight of M. Assume that there exists a sequence $\{u_n\}$ of unitaries of $\tilde{M} \rtimes_\theta \mathbf{R}$ with $\mathrm{Ad}u_n \circ \tilde{\tilde{\rho}} \to \tilde{\tilde{\sigma}}$ as $n \to \infty$. Then there exists a sequence $\{v_n\}$ of unitaries of M with $\mathrm{Ad}v_n \circ \rho \to \sigma$.*

Proof. Since (φ, ρ) and (φ, σ) are invariant pairs, it is possible to identify $\tilde{\tilde{\rho}}$ with $\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})}$ and $\tilde{\tilde{\sigma}}$ with $\sigma \otimes \mathrm{id}_{B(L^2\mathbf{R})}$ through Takesaki duality, respectively (It is possible to choose the same identification between $M \otimes B(L^2\mathbf{R})$ and $\tilde{M} \rtimes_\theta \mathbf{R}$ for $\tilde{\tilde{\rho}}$ and $\tilde{\tilde{\sigma}}$. See the argument preceding to Lemma 3.10 of Masuda–Tomatsu [15]). Then by (the proof of) Lemma 3.11 of Masuda–Tomatsu [15], there exist an isomorphism π from $M \otimes B(L^2\mathbf{R})$ to M and unitaries u_ρ, u_σ of M satisfying
$$\pi \circ (\rho \otimes \mathrm{id}) \circ \pi^{-1} = \mathrm{Ad}u_\rho \circ \rho,$$

$$\pi \circ (\sigma \otimes \mathrm{id}) \circ \pi^{-1} = \mathrm{Ad} u_\sigma \circ \sigma$$

(Although in the statement of Lemma 3.11 of Masuda–TOmatsu [15], the isomorphism π depends on the choice of ρ, π turns out to be independent of ρ by its proof). Then we have

$$\begin{aligned}
&\mathrm{Ad}(u_\sigma^* \pi(u_n) u_\rho) \circ \rho \\
&= \mathrm{Ad}(u_\sigma^* \pi(u_n)) \circ \pi \circ (\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})}) \circ \pi^{-1} \\
&= \mathrm{Ad} u_\sigma^* \circ \pi \circ (\mathrm{Ad} u_n \circ (\rho \otimes \mathrm{id}_{B(L^2\mathbf{R})})) \circ \pi^{-1} \\
&\to \mathrm{Ad} u_\sigma^* \circ \pi \circ (\sigma \otimes \mathrm{id}_{B(L^2\mathbf{R})}) \circ \pi^{-1} \\
&= \mathrm{Ad} u_\sigma^* \circ (\mathrm{Ad} u_\sigma \circ \sigma) \\
&= \sigma.
\end{aligned}$$

□

Lemma 20. *Let ρ be an endomorphism with finite index and with (φ, ρ) an invariant pair. Let $E_{\tilde{\tilde{\rho}}}$ be the minimal expectation from $\tilde{\tilde{M}}$ to $\tilde{\tilde{\rho}}(\tilde{\tilde{M}})$. Then we have the following.*

(1) *For each $x \in \tilde{M}$, we have $E_{\tilde{\tilde{\rho}}}(x) = E_{\tilde{\rho}}(x)$.*

(2) *For any $s \in \mathbf{R}$, we have $E_{\tilde{\tilde{\rho}}}(\lambda_t^\theta) = \lambda_t^\theta$.*

Proof. This is shown in the proof of Theorem 4.1 of Longo [13]. □

Lemma 21. *For $\xi \in L^2(\mathbf{R}, \tilde{M})$, set*

$$V_{\tilde{\tilde{\rho}}}(\xi)(s) := V_{\tilde{\rho}}(\xi(s)).$$

Then $V_{\tilde{\tilde{\rho}}}$ is an isometry of $L^2(\mathrm{R}, \tilde{M})$ satisfying

$$V_{\tilde{\tilde{\rho}}}^* x V_{\tilde{\tilde{\rho}}} = \phi_{\tilde{\tilde{\rho}}}(x)$$

for any $x \in M$, where $\phi_{\tilde{\tilde{\rho}}} = \tilde{\tilde{\rho}}^{-1} \circ E_{\tilde{\tilde{\rho}}}$.

Proof. The first statement is shown by the following computation.

$$\begin{aligned}
\|V_{\tilde{\tilde{\rho}}}(\xi)\|^2 &= \int_{\mathbf{R}} \|V_{\tilde{\rho}}(\xi(s))\|^2 \, d\mu(s) \\
&= \int_{\mathbf{R}} \|\xi(s)\|^2 \, d\mu(s) \\
&= \|\xi\|^2
\end{aligned}$$

for $\xi \in L^2(\mathbf{R}, \tilde{M})$. Next, we show the latter statement. Choose $x \in M$ and $\xi \in L^2(\mathbf{R}, \tilde{M})$. Then we have

$$\begin{aligned}
V_{\tilde{\tilde{\rho}}}^* \circ \pi_\theta(x) \circ V_{\tilde{\tilde{\rho}}}(\xi) &= V_{\tilde{\tilde{\rho}}}^* \pi_\theta(x)(s \mapsto V_{\tilde{\rho}}(\xi(s))) \\
&= V_{\tilde{\tilde{\rho}}}^* (s \mapsto \theta_{-s}(x) \circ V_{\tilde{\rho}}(\xi(s))) \\
&= (s \mapsto V_{\tilde{\rho}}^* \circ \theta_{-s}(x) \circ V_{\tilde{\rho}}(\xi(s))) \\
&= (s \mapsto \phi_{\tilde{\rho}}(\theta_{-s}(x))(\xi(s))) \\
&= (s \mapsto \theta_{-s}(\phi_{\tilde{\rho}}(x))(\xi(s))) \\
&= \pi_\theta(\phi_{\tilde{\rho}}(x))(\xi) \\
&= \phi_{\tilde{\tilde{\rho}}}(\pi_\theta(x))(\xi).
\end{aligned}$$

In order to show the fourth equality of the above, we used Lemma 18. The last equality of the above follows from Lemma 20. For $t \in \mathbf{R}$ and $\xi \in L^2(\mathbf{R}, \tilde{M})$, we have

$$\begin{aligned} V_{\tilde{\tilde{\rho}}}^* \lambda_t^\theta V_{\tilde{\tilde{\rho}}} \xi &= V_{\tilde{\tilde{\rho}}}^* (s \mapsto V_{\tilde{\rho}}(\xi(s-t)) \\ &= s \mapsto V_{\tilde{\rho}}^* V_{\tilde{\rho}}(\xi(s-t)) \\ &= \lambda_t^\theta(\xi). \end{aligned}$$

Thus we are done. □

Lemma 22. *Let N be a von Neumann algebra and $\{V_n\}_{n=0}^\infty$ be a sequence of isometries on the standard Hilbert space $L^2(N)$ such that for each n, the map $\Phi_n : N \ni x \mapsto V_n^* x V_n$ is a left inverse of an endomorphism of N. Then the following two conditions are equivalent.*

(1) *The sequence of operators $\{V_n\}_{n=1}^\infty$ converges to V_0 strongly.*
(2) *We have $\|\psi \circ \Phi_n - \psi \circ \Phi_0\| \to 0$ for any $\psi \in N_*$.*

Proof. This is shown by the same argument as that of the proof of Lemma 3.3 of Masuda–Tomatsu [15]. □

Lemma 23. *Let $\{u_n\}$ be a sequence of unitaries of $\tilde{M}$ satisfying the following conditions.*

(1) *We have $\mathrm{Ad}u_n \circ \tilde{\rho} \to \tilde{\sigma}$ as $n \to \infty$.*
(2) *For any compact subset F of $\mathbf{R}$, we have $\theta_t(u_n) - u_n \to 0$ uniformly for $t \in F$.*

Then we have $\mathrm{Ad}u_n \circ \rho \to \sigma$.

Proof. By Lemmas 19, 21 and 22, it is enough to show that $V_{\tilde{\tilde{\rho}}} u_n^* \to V_{\tilde{\tilde{\sigma}}}$. Notice that we have

$$\begin{aligned} &V_{\tilde{\tilde{\rho}}} u_n^* (\xi \otimes f) \\ &= (s \mapsto V_{\tilde{\rho}}(\theta_{-s}(u_n^*)(\xi)) f(s)) \end{aligned}$$

for any $\xi \in L^2(M)$ and $f \in L^2\mathbf{R}$. Hence we have

$$\begin{aligned} &\|V_{\tilde{\tilde{\rho}}} u_n^* (\xi \otimes f) - V_{\tilde{\tilde{\sigma}}}(\xi \otimes f)\|^2 \\ &= \int_{\mathbf{R}} \|V_{\tilde{\rho}}(\theta_{-s}(u_n^*)(\xi)) - V_{\tilde{\sigma}}(\xi)\|^2 |f(s)|^2 \, ds \\ &\leq \int_{\mathbf{R}} \|(V_{\tilde{\rho}}((\theta_{-s}(u_n^*) - u_n^*)(\xi))\|^2 |f(s)|^2 \, ds + \int_{\mathbf{R}} \|V_{\tilde{\rho}}(u_n^*(\xi)) - V_{\tilde{\sigma}}(\xi)\|^2 |f(s)|^2 \, ds \\ &\to 0 \end{aligned}$$

by the Lebesgue dominant convergence theorem. Note that in order to show the last convergence, we use Lemmas 17, 18 and 22. □

7. The proof of the main theorem

Lemma 24. *Let M be an AFD factor and σ be a finite index endomorphism of M with $d(\sigma) = d$. Then there exists an endomorphism λ with the following properties.*

(1) *The endomorphism λ is approximately inner.*
(2) *We have $d(\lambda) = d$.*

(3) *The endomorphism λ has Connes–Takesaki module and it is $\theta_{-\log d}|_{\mathcal{Z}(\tilde{M})}$.*

Proof. By the proof of Theorem 3 of Kosaki–Longo [9], there exists an endomorphism λ_0 of the AFD factor of type II_1 with $d(\lambda_0) = d$. Then $\mathrm{id}_M \otimes \lambda_0$ is an endomorphism of M with $d(\mathrm{id} \otimes \lambda_0) = d$ and with $\mathrm{mod}(\mathrm{id} \otimes \lambda_0)$ trivial. Hence by the existence of a right inverse of the Connes–Takesaki module of automorphisms (See Sutherland–Takesaki [23]), there exists an automorphism α of M with $\mathrm{mod}(\alpha \circ \lambda_0) = \theta_{-\log(d)}$. By Theorem 3.15 of Masuda–Tomatsu (or by the same argument of our paper), it is shown that $\lambda := \alpha \circ \lambda_0$ is approximately inner. □

Now, we return to the proof of the main theorem.

Proof of implication (1) $\Rightarrow$ (2) *of Theorem 2.* Let ρ, σ be endomorphisms of $\mathrm{End}(M)_0$ with the first condition of Theorem 2. Then by Lemma 24, there exist endomorphisms $\lambda, \mu \in \mathrm{End}(M)_0$ with the following properties.

(1) We have $d(\lambda) = d(\sigma)$, $d(\mu) = d(\rho)$.
(2) We have $\tilde{\lambda}|_{\mathcal{Z}(\tilde{M})} = \theta_{-\log(d(\sigma))}|_{\mathcal{Z}(\tilde{M})}$ and $\tilde{\mu}|_{\mathcal{Z}(\tilde{M})} = \theta_{-\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})}$.
(3) The endomorphisms λ and μ are approximately inner.

By the second condition, we have

$$\begin{aligned}\phi_{\tilde{\rho}} \circ \phi_{\tilde{\lambda}}|_{\mathcal{Z}(\tilde{M})} &= \phi_{\tilde{\rho}} \circ \theta_{\log d(\sigma)}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \theta_{-\log(d(\sigma)/d(\rho))} \circ \theta_{\log d(\sigma)}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \theta_{\log(d(\rho))}|_{\mathcal{Z}(\tilde{M})} \\ &= \phi_{\tilde{\sigma}} \circ \phi_{\tilde{\mu}}|_{\mathcal{Z}(\tilde{M})}.\end{aligned}$$

Hence by replacing ρ by $\lambda \circ \rho$ and σ by $\mu \circ \sigma$ respectively, we may assume that $d(\rho) = d(\lambda)$ and $\phi_{\tilde{\rho}}|_{\mathcal{Z}(M)} = \phi_{\tilde{\sigma}}|_{\mathcal{Z}(M)}$. By Proposition 17, there exists a sequence $\{u_n\}$ of unitaries of $\tilde{M}$ satisfying the assumptions of Lemma 23. Hence by Lemma 23, we have $\mathrm{Ad}u_n \circ \rho \to \sigma$. □

8. Appendix (A proof of the characterization of central triviality of automorphisms of the AFD factors)

In this section, we will see that it is possible to give a proof of a characterization theorem of central triviality of automorphisms of the AFD factors by a similar strategy to the proof of Theorem 2, which is independent of the types of the AFD factors.

Let M be an AFD factor of type III. Let α be an automorphism of M and $\tilde{\alpha}$ be its canonical extension. Set

$$p := \min\{q \in \mathbf{N} \mid \tilde{\alpha}^q \text{ is centrally trivial}\},$$
$$G := \mathbf{Z}/p\mathbf{Z}.$$

Lemma 25. *The action $\{\tilde{\alpha}_n \circ \theta_t\}_{(n,t) \in G \times \mathbf{R}}$ of $G \times \mathbf{R}$ on $\tilde{M}_{\omega,\theta}$ is faithful.*

Proof. We will show this lemma by contradiction. Let φ be a normal faithful state of $\tilde{M}$ and $\{\psi_j\}_{j=1}^{\infty}$ be a norm dense sequence of the unit ball of $\tilde{M}_*$. Assume that there existed a pair $(n,t) \in (G \times \mathbf{R}) \setminus \{(0,0)\}$ satisfying $\tilde{\alpha}_n \circ \theta_{-t}(a) = a$ for any $a \in \tilde{M}_{\omega,\theta}$. Then the automorphism $\tilde{\alpha}_n \circ \theta_{-t}$ would be centrally non-trivial because $\tilde{\alpha}_n \circ \theta_{-t}$ is trace-scaling if $t \neq 0$. Hence there would exist an x of $\tilde{M}_\omega$, which can

never be of $\tilde{M}_{\omega,\theta}$, with $\tilde{\alpha}_n(x) \neq \theta_t(x)$ and with $\|x\| \leq 1$. Take a representing sequence $\{x_k\}$ of x with $\|x_k\| \leq 1$ for any k. Then we would have

$$\begin{aligned}
&\lim_{k\to\omega} \|\tilde{\alpha}_n(x_k) - \theta_t(x_k)\|^\sharp_{\varphi\circ\theta_s} \\
&= \text{weak} \lim_{k\to\omega} \frac{1}{2}(|\tilde{\alpha}_n(x_k) - \theta_t(x_k)|^2 + |(\tilde{\alpha}_n(x_k) - \theta_t(x_k))^*|^2) \\
&= 2\delta > 0
\end{aligned}$$

for some $\delta > 0$ (The constant δ does not depend on the choice of $s \in \mathbf{R}$). Then for each natural number L, there would exist $k \in \mathbf{N}$ satisfying the following three conditions.

(1) We have $\|x_k\| \leq 1$.
(2) We have

$$\|\theta_t(x_k)\psi_j - \psi_j\theta_t(x_k)\|(= \|x_k(\psi_j \circ \theta_t) - (\psi_j \circ \theta_t)x_k\|) < \frac{1}{L}$$

for $j = 1, \cdots, L$, $|t| \leq L$ (Use the compactness of $\{\psi_j \circ \theta_t \mid t \in L\}$. See also the argument just after Lemma 11).
(3) We have

$$\|\tilde{\alpha}_n(x_k) - \theta_t(x_k)\|^\sharp_\varphi > \delta.$$

Let $\Theta : L^\infty([-L, L], dm(t)) \otimes (\tilde{M}, \varphi) \to (\tilde{M}_{\omega,\theta}, \varphi^\omega)$ be the inclusion mentioned in Section 5 (an inclusion coming from the Rohlin property of θ), where $dm(t)$ is the normalized Haar measure of $[-L, L]$. Set

$$\tilde{y} := ([-L, L] \ni s \mapsto \theta_s(x_k)) \in L^\infty([-L, L], dm(s)) \otimes \tilde{M},$$

$$y := \Theta(\tilde{y}).$$

Since we would have $\tilde{\alpha}_n \circ \theta_{-t}$ is trivial on $\tilde{M}_{\omega,\theta}$, we would have

$$(\tilde{\alpha}_n(\Theta(f)))_s = \tilde{\alpha}_n(\Theta(f)_{s-t})$$

for $f \in L^\infty([-L, L]) \otimes \tilde{M}$ and $s \in [-L + t, L - t]$, where f_s is the evaluation of the function f at $s \in [-L, L]$. Hence we would have

$$\begin{aligned}
\|\tilde{\alpha}_n(y) - y\|^\sharp_{\varphi^\omega} &\geq (\int_{[-L+t,L-t]} (\|\tilde{\alpha}_n(\theta_{s-t}(x_k)) - \theta_s(x_k)\|^\sharp_\varphi)^2 \, ds \\
&\quad - \int_{[-L,-L+t]\cup[L-t,L]} 2^2 \, ds)^{1/2} \\
&\geq (\int_{[-L,L]} \delta^2 \, ds - \frac{4t}{L})^{1/2} \\
&= (\delta^2 - \frac{4t}{L})^{1/2}.
\end{aligned}$$

Since we have

$$(\theta_r(y))_s = \theta_s(y)$$

for any $0 < r < 1$, $s \in [-L+r, L-r]$, we have

$$\begin{aligned}\|\theta_r(y) - y\|_{\varphi^\omega}^\sharp &= \Big(\int_{[-L,L]} (\|(\theta_r(y))_s - y_s\|_\varphi^\sharp)^2 \, ds\Big)^{1/2} \\ &\leq \Big(\int_{[-L,-L+1]\cup[L-1,L]} 2^2 \, ds\Big)^{1/2} \\ &= \frac{2}{\sqrt{L}}\end{aligned}$$

for $|r| \leq 1$. We also have

$$\begin{aligned}\|[y, \psi_j]\| &= \|[y, \psi_j]|_{\Theta(\mathbf{C}\otimes\tilde{M})}\| \\ &\leq \|[y, \psi_i]|_{\Theta(L^\infty([-L,L])\otimes\tilde{M})}\| \\ &= \int_{[-L,L]} \|[\tilde{y}_s, \psi_i]\| \, ds \\ &= \int_{[-L,L]} \|[\theta_s(x_k), \psi_j]\| \, ds \\ &< \int_{[-L,L]} \frac{1}{L} \, ds \\ &= \frac{1}{L}\end{aligned}$$

for $j = 1, \cdots, L$. Hence there would exist a sequence $\{y_l\}$ of $\tilde{M}$ with the following properties.

(1) We have $\|y_l\| \leq 1$.
(2) We have $\|[y_l, \psi_j]\| \to 0$ for any $j = 1, 2, \cdots$.
(3) For any $j = 1, 2, \cdots$, we have $\|\theta_r(y_l) - y_l\|_\varphi^\sharp \to 0$ uniformly for $s \in [-1, 1]$.
(4) We have $\|\tilde{\alpha}_n(y_l) - \theta_t(y_l)\|_\varphi^\sharp \geq \delta/2$ for any l.

This would contradict the assumption that $\tilde{\alpha}_n \circ \theta_{-t}$ were trivial on $\tilde{M}_{\omega,\theta}$. □

Lemma 26. *For each $p \in (G \hat{\times} \mathbf{R}) = \hat{G} \otimes \mathbf{R}$, there exists a unitary u of $\tilde{M}_{\omega,\theta}$ with $\tilde{\alpha}_n \circ \theta_t(u) = \langle (n,t), p \rangle u$ for any $(n,t) \in G \times \mathbf{R}$.*

Proof. The proofs of Theorems 4.10 and 7.7 of Masuda–Tomatsu [19] works in our case. □

Lemma 27. *There exist a non-zero projection e of $(\tilde{M}_{\omega,\theta})^\theta$ with $\tilde{\alpha}(e)$ orthogonal to e.*

Proof. By the previous lemma, for each natural number l, there exists a unitary u of $\tilde{M}_{\omega,\theta}$ with $\tilde{\alpha}(u) = e^{2\pi i/p} u$ and with $\theta_t(u) = e^{-it/l} u$ for any t. Then there exists a spectral projection e of u with $\tilde{\alpha}(e) \leq 1 - e$, $\tau^\omega(e) = 1/p$ and with $\tau^\omega(e - \theta_t(e)) \leq 1/(2l)$ for $|t| \leq 1$. By the usual diagonal argument, it is possible to choose a desired projection. □

Theorem 28. (See Theorem 1 (2) of Kawahigashi–Sutherland–Takesaki) *For an automorphism α of M, α is centrally trivial if and only if its canonical extension is inner.*

Proof. First, assume that $\tilde{\alpha}$ is not centrally trivial. Then by the previous lemma, neither is $\tilde{\tilde{\alpha}}$. Hence neither is α centrally trivial (See, for example, Lemmas 5.11 and 5.12 of Sutherland–Takesaki [22]). The above argument means that if α is centrally trivial, then $\tilde{\alpha}$ is centrally trivial. Since $\tilde{M}$ is of type II, any centrally trivial automorphism of $\tilde{M}$ is inner. The opposite direction is trivial by the central triviality of a modular endomorphism group. $\square$

Remark 29. Finally, we remark that by our results and the result of Masuda [14], if we admit that the AFD factors are completely classified by their flows of weights, it is possible to classify the actions of discrete amenable groups on the AFD factors without separating cases by the types of the factors.

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